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# Self-adjoint extensions of the Hamiltonian for a charged spin-$\frac{1}{2}$ particle in the Aharonov–Bohm field

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**Abstract.** Recently we studied self-adjoint extensions (SAEs) of the Aharonov–Bohm Hamiltonian for a charged Schrödinger particle of spin 0. In this paper we discuss the relation between the SAEs for a spinless particle and those for a spin-$\frac{1}{2}$ particle in the same environment. We also consider SAEs when a Coulomb potential is added. We furthermore clarify a few confusing issues which have been discussed in some recent papers regarding the relativistic Dirac particle in the Aharonov–Bohm field.

## 1. Introduction

This paper is a sequel to two earlier papers [1, 2]. In [1] we examined self-adjoint extensions (SAEs) for the Aharonov–Bohm Hamiltonian, i.e. the non-relativistic Schrödinger Hamiltonian for a spinless charged particle in the presence of an infinitely thin thread of magnetic flux [3]. In [2] we discussed the same problem but for the Dirac equation. As we noted in [1], there is a vast literature on the subject of SAEs of the quantum mechanical Hamiltonian and various rather sophisticated methods are available [4].

One can tell whether or not SAEs for a given Hamiltonian are possible by finding the 'deficiency index' [4]. For the Schrödinger Hamiltonians that we are going to discuss in the following, it can be shown that the deficiency index is (1,1) for a relevant partial wave state, which means that there is a one-parameter family of SAEs, i.e. a one-parameter family of different dynamics. Rather than using the deficiency index, however, we prefer to follow the pedestrian approach which we took in [1]. This can be summarized as follows. We assume an *attractive* square well potential of radius $R$ and depth $D$ acting in a relevant angular momentum state and solve the Schrödinger equation in the usual manner. Assume that there is a bound state of energy $-\hbar\kappa^2/2m$, where $m$ is the mass of the particle. We take the limit $R \to 0$ so that the square well potential becomes an attractive $\delta$-function potential. In taking the limit we fine-tune the depth $D$ in such a way that the bound state energy remains the same. The ensuing $\delta$-function potential is characterized by parameter $\kappa$. The wavefunction in this limit is singular at the origin but is normalizable provided that the magnetic flux is within a certain range. The assumption of the existence of the bound state is not essential. If there is no bound state, the $\delta$-function can be characterized by a quantity related to scattering. In [2] we applied essentially the same method to the Dirac equation.

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The present paper was motivated by very recent papers by Hagen [5] and by Bordag and Voropaev [6] in which they examined the Aharonov–Bohm problem for a spin-$\frac{1}{2}$ Schrödinger particle. The main purpose of the present paper is to examine the relation between the results of [5, 6] and those of [1]. Hagen also examined the case in which, in addition to the magnetic flux, there is an additional Coulomb interaction acting on the particle [5]. We amend Hagen's result and give a complete prescription for this case. We then go on to discuss the relativistic version of the same problem in which the Schrödinger equation is replaced by the Dirac equation. There were a few rather confusing points in [2] which Hagen criticized [5]. We will clarify them.

In section 2 we summarize and extend the results obtained in [1]. In section 3 we discuss how a spin-$\frac{1}{2}$ Schrödinger particle can be accommodated in the framework of [1]. In section 4 we examine the case in which a Coulomb potential is added. In section 5 we examine some related aspects of the Dirac version of the problem. Discussions are given in section 6.

## 2. Spinless Schrödinger particle

For clarity and to establish notation let us summarize relevant parts of [1]. Consider the Aharonov–Bohm Hamiltonian, i.e. the Schrödinger Hamiltonian for a spinless charged particle in the presence of an infinitely thin thread of magnetic flux at the origin:

$$H_0 = \frac{\hbar^2}{2m}\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\nu^2}{r^2}\right) \tag{2.1}$$

where $r = (x^2 + y^2)^{1/2}$. $\nu$ is a constant related to the magnetic flux $\Phi$ and the (integral) angular momentum $n$ through

$$\nu = n + \alpha \qquad \alpha = -e\Phi/(2\pi\hbar) \tag{2.2}$$

where $e$ is the charge of the particle. The Hamiltonian $H_0$ of (2.1) is not well defined unless the boundary condition on the wavefunction at the origin is specified. As shown in [1], when $|\nu| < 1$, there is a one-parameter family of SAEs of $H_0$. For a given value of $\alpha$, there are two adjacent integers for $n$ such that $|\nu| < 1$. Since $\nu$ appears as $\nu^2$ in $H_0$, we can assume that $\nu > 0$, and we do so in the following except in discussing the Dirac equation in section 5.

Let us first replace the infinitely thin magnetic flux with a uniform flux of finite radius $R$. In addition, we assume a 'non-gauge' potential $V(r)$ which, together with the magnetic interaction, forms an attractive square well potential of radius $R$ and depth $D$. It is understood that there is no singularity at the origin and the wavefunction satisfies the usual boundary condition everywhere. For $r < R$ the radial part of the Schrödinger equation becomes

$$\left[\frac{\hbar^2}{2m}\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{n^2}{r^2}\right) - D\right]\phi = E\phi \qquad \text{for } r < R \tag{2.3}$$

while $H_0\phi = E\phi$ remains to apply for $r > R$. Assume that there is one and only one bound state. The $\phi(r)$ for the bound state is given by the Bessel functions, $J_n(k_0 r)$ for $r < R$ and

$K_\nu(\kappa r)$ for $r > R$, where $k_0^2 = 2m(D+E)/\hbar^2$, $\kappa^2 = -2mE/\hbar^2$ and suffix $n$ of $J_n$ stands for $|n|$. The eigenvalue $E$ can be determined from the matching condition at $r = R$:

$$F(k_0 R) = G(\kappa R) \tag{2.4}$$

$$F(k_0 R) = \left.\frac{r}{\phi}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right|_{r=R-0} \qquad G(\kappa R) = \left.\frac{r}{\phi}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right|_{r=R+0}. \tag{2.5}$$

We now let $R \to 0$, but keep $\kappa$ fixed at a finite value which we can choose as we like. Then $D \to \infty$ and the square well potential becomes an attractive $\delta$-function potential which is characterized by the parameter $\kappa$. If $\kappa R \ll 1$, we obtain

$$G(\kappa R) \simeq -\nu - \frac{2\Gamma(1-\nu)}{\Gamma(\nu)}\left(\frac{\kappa R}{2}\right)^{2\nu} \tag{2.6}$$

which we will find useful later. Once the $\delta$-function potential is specified, $H_0$ of (2.1) is completely defined. Although we assume a square-well potential for $r < R$, the details of the potential for $r < R$ becomes unimportant in the limit of $R \to 0$. After the limit is taken, the bound state wavefunction is given for the entire range of $r$ by $K_\nu(\kappa r)$, which is singular at the origin but is square integrable.

The $\phi(r)$ for the scattering problem for $r > R$ can be written as

$$\phi(r) = J_\nu(kr)\cos\eta - N_\nu(kr)\sin\eta \tag{2.7}$$

where $k^2 = 2mE/\hbar^2$. $N_\nu(kr)$ is singular at the origin but is locally square integrable. In the classic case of Aharonov and Bohm, they assumed the standard boundary condition which requires that the wavefunction be regular at the origin [3]. Hence $\eta = 0$ in the Aharonov–Bohm case. There is no bound state.

Returning to the $\phi(r)$ of (2.7), we obtain for $kR \ll 1$

$$G(kR) = \left.\frac{r}{\phi}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right|_{r=R+0} \simeq -\nu + \frac{2\pi}{\Gamma^2(\nu)}\left(\frac{kR}{2}\right)^{2\nu}\cot\eta. \tag{2.8}$$

Here it is understood that $\eta \neq 0$. In the limit of $R \to 0$, this $G(kR)$ can be equated with the $G(\kappa R)$ of (2.6) for the bound state. Note that the depth $D$ of the potential inside the flux becomes infinite in this limit and hence $F(k_0 R)$ becomes *energy independent*. In setting the boundary condition $G(\kappa R) = F(k_0 R) = G(kR)$, $F(k_0 R)$ plays only an intermediary role. We thus obtain

$$\tan\eta(k) = -\sin(\nu\pi)(k/\kappa)^{2\nu}. \tag{2.9}$$

In the special case of $\kappa = 0$ (with a 'zero mode'), $\eta$ can be determined by taking the limit of $\kappa \to 0$ in (2.9) with the understanding that $k^2 > 0$. This leads to $\eta(k) = \pi/2$ for any value of $k$. In this case (2.7) becomes $\phi(r) = -N_\nu(kr)$. As we will see in the next section, this is the situation realized in Hagen's model [5,7] and also in Bordag and Voropaev's model with the gyromagnetic ratio $g = 2$ [6].

In the absence of the magnetic flux ($\alpha = 0$, $\nu = n = 0$), for any SAE of $H_0$ (such that the wavefunction is singular at the origin), there is a bound state. This is because any additional attractive potential in two dimensions, no matter how weak, produces a bound state. Therefore all possible SAEs in the absence of the magnetic flux can be parametrized in terms of $\kappa$. In the presence of the magnetic flux, however, there are SAEs such that there is no bound state. This is possible because the repulsive term $\nu^2/r^2$ of the Hamiltonian counteracts the additional attractive non-gauge potential. In order to include such a situation it is convenient to parametrize the SAEs, for example, in terms of

$$\lambda \equiv -\sin(\nu\pi) \lim_{k\to 0}[k^{2\nu} \cot \eta(k)]. \tag{2.10}$$

If $\lambda > 0$, there is a bound state and $\lambda$ is related to $\kappa$ by $\lambda = \kappa^{2\nu}$. If $\lambda < 0$, there is no bound state. In this case (2.9) is replaced by

$$\tan \eta(k) = \sin(\nu\pi)(k^{2\nu}/|\lambda|). \tag{2.11}$$

## 3. Spin-$\frac{1}{2}$ Schrödinger particle

In this section we discuss how the above results accommodate the spin-$\frac{1}{2}$ Schrödinger particle. For a charged particle of spin-$\frac{1}{2}$ in the presence of an infinitely thin magnetic flux, the Hamiltonian becomes

$$H = H_0 + \alpha s \delta(r)/r \tag{3.1}$$

where $s = \pm 1$ for spin 'up' and spin 'down', respectively. We are interested in the case $\alpha s < 0$. Hagen [5, 7] examined this problem by assuming that the magnetic field is concentrated on a ring of radius $R$ and the spin interaction term of (3.1) is replaced by

$$\alpha s \delta(r - R)/R. \tag{3.2}$$

He did not assume any additional non-gauge potential within radius $R$. By taking the limit $R \to 0$, Hagen found that the wavefunction for $\nu < 1$ ($|m + \alpha| < 1$ in his notation) can be irregular, i.e. singular at the origin. This wavefunction represents a zero-mode, i.e. a bound state with zero binding energy, and it precisely corresponds to the SAE with $\kappa = 0$ for a spinless particle as we touched upon earlier below (2.9). This zero-mode is one of the zero modes found by Aharonov and Casher under a more general condition [8]. Hagen said that there is no bound state, which is correct in the sense that the zero mode is not quite a bound state. For the scattering wavefunction, we find that $\eta(k) = \pi/2$ in the way as we explained earlier below (2.9). The phrase 'non-gauge' potential that we have been using may be somewhat confusing. By this we meant in section 2 a potential other than that due to the magnetic field. However, the non-gauge potential that leads to the SAE with $\kappa = 0$ for the spinless particle of section 2 is equivalent to the spin interaction of (3.2) without any additional non-gauge potential.

Bordag and Voropaev considered a Schrödinger particle of spin $\frac{1}{2}$ with an arbitrary gyromagnetic ratio $g$ [6]. They assumed no additional non-gauge interaction. If $g = 2$, their model becomes identical with Hagen's model in the limit $R \to 0$, it corresponds to the SAE of $H_0$ of (2.1) with $\kappa = 0$. If $g \neq 2$, the particle has an anomalous magnetic moment. Bordag and Voropaev found that there is no bound state if $g < 2$. If $g > 2$, the energy of the bound state (with $\nu < 1$ as we are assuming) diverges in the limit $R \to 0$. They pointed out that the binding energy can be made finite in this case by lettting $g \to 2$ as $R \to 0$. This situation can be accommodated within the scheme discussed in [1] with

$\kappa \neq 0$. The real electron has an anomalous magnetic moment such that $g$ is slightly greater than 2. In this sense the limit of $g \to 2$ is artificial. On the other hand, if the radius $R$ of the thread is finite but small, there will be a bound state which is almost singular at the origin. This situation can be well simulated by an appropriate SAE discussed in [1]. Bordag and Voropaev discussed this interesting aspect of the problem in detail. They further considered situations in which there are more than one bound state.

Although unrealistic for the electron, the case of $g < 2$ is of some interest. Here it is understood that $g > 0$. (The situation with $g < 0$ can be obtained by reversing the spin direction.) Let us examine $\eta$ of the scattering wavefunction. Let us consider the spin term of the surface interaction type (3.2), i.e.

$$\left(\frac{g}{2}\right) \alpha s \delta(r - R)/R \tag{3.3}$$

where it is understood that $\alpha s < 0$. This is one of the three models that Bordag and Voropaev considered for the magnetic field distribution within the flux. Following them, we assume no additional non-gauge potential so that the potential for $r < R$ is zero. Then the wavefunction for $r < R$ is given by $J_n(kR)$, and hence

$$F(kR) \simeq |n| - \frac{(kR)^2}{2(|n|+1)} - \frac{g}{2}\alpha s \tag{3.4}$$

where the last term is due to the spin interaction of (3.3). If we equate this with $G(kR)$, remembering that $0 < \nu \leqslant 1$, we find

$$\nu + |n| - \frac{g}{2}\alpha s \simeq \frac{2\pi}{\Gamma^2(\nu)} \left(\frac{kR}{2}\right)^{2\nu} \cot\eta. \tag{3.5}$$

For $g \neq 2$, the left-hand side is a non-zero constant. This means that, for any $k > 0$, $\cot\eta \to \infty$ and $\eta \to 0$ as $R \to 0$. For $g < 2$, therefore, irregular solutions are not allowed.

Let us add that, if we assume a non-gauge potential for $r < R$ as we did in [1], we can have SAEs with irregular solutions. In this case the SAEs can be specified in terms of the parameter $\lambda (< 0)$ of (2.10). The boundary condition at $r = R$ in this case reads as

$$F(k_0 R) \simeq \frac{2\pi}{\Gamma^2(\nu)} \left(\frac{kR}{2}\right)^{2\nu} \cot\eta. \tag{3.6}$$

For a given value of $k$, (3.6) defines $k_0$ as a function of $R$. When $R \to 0$, $k_0^2 \simeq 2mD/\hbar^2$ where $D$ is a function of $R$. An SAE can be obtained by letting $R \to 0$ but keeping the value of $\eta(k)$ fixed. We can choose the 'reference value' of $k$ as we like. If we want to specify the SAE in terms of $\lambda$ of (2.10), we choose $k = +0$.

## 4. When a Coulomb interaction is added

Hagen considered the case in which, in addition to the magnetic flux, there is a Coulomb potential acting on the particle [5]. He claimed that SAEs are possible, i.e. irregular solutions are admissible, only if $\nu < \frac{1}{2}$. We will show that, although there is an intriguing difference between the cases with $\nu < \frac{1}{2}$ and $\frac{1}{2} < \nu < 1$, SAEs are also possible for $\frac{1}{2} < \nu < 1$.

The Hamiltonian that we consider is

$$H = H_0 + \frac{\hbar^2}{2m}\frac{\xi}{r} = \frac{\hbar^2}{2m}\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\nu^2}{r^2} + \frac{\xi}{r}\right) \tag{4.1}$$

where $\xi$ may be positive or negative. The $\xi$ of [5] corresponds to our $(\hbar^2/2m)\xi$. Let us examine SAEs in the same manner as we did in [1]. We assume an *attractive* non-gauge potential of radius $R$. After solving the Schrödinger equation in the usual manner, we let $R \to 0$ so that the square well potential becomes an attractive $\delta$-function potential. In (4.1) we did not include the spin term of (3.3). This is because, as we discussed in section 3, such a spin interaction can be thought of as a special SAE (with $\kappa = 0$) of $H_0$.

Let us examine possible bound states. The wavefunction for the bound state of energy $E = -\hbar^2\kappa^2/2m$ takes the following form:

$$\phi(r) = (\kappa r)^{-\nu} \mathrm{e}^{-\kappa r} U(a, b, 2\kappa r) \tag{4.2}$$

$$a = \frac{1}{2}\left(1 - 2\nu + \frac{\xi}{\kappa}\right) \qquad b = 1 - 2\nu \tag{4.3}$$

where $U(a, b, 2\kappa r)$ is the Kummer function [9]. The Kummer function is a linear combination of regular and irregular solutions of Kummer's equation such that it converges as $r \to \infty$. It is generally singular at the origin.

We now calculate $G(\kappa R)$ of (2.5) but for the $\phi(r)$ of (4.2). In doing so the following formulae are useful. For $U(a, b, z)$ with $|z| \ll 1$, we have

$$\frac{z}{U}\frac{\mathrm{d}U}{\mathrm{d}z} \simeq \frac{\Gamma(b)\Gamma(1+a-b)}{\Gamma(a)\Gamma(1-b)} z^{1-b} \qquad \text{for } 0 < b < 1 \tag{4.4}$$

$$\frac{z}{U}\frac{\mathrm{d}U}{\mathrm{d}z} \simeq -\frac{a}{b} z + \frac{\Gamma(b)\Gamma(1+a-b)}{\Gamma(a)\Gamma(1-b)} z^{1-b} \qquad \text{for } -1 < b < 0. \tag{4.5}$$

The two regions regarding $b$ of (4.4) and (4.5) correspond to $\nu < \frac{1}{2}$ and $\frac{1}{2} < \nu < 1$, respectively. We can now write down $G(\kappa R)$:

$$G(\kappa R) = \left.\frac{r}{\phi}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right|_{r=R+0} \simeq -\nu + \frac{\xi R}{2\nu - 1}\theta(\nu - \tfrac{1}{2}) + \frac{\Gamma(1-2\nu)}{\Gamma(2\nu)} \frac{\Gamma\left(\frac{1}{2} + \nu + \frac{\xi}{2\kappa}\right)}{\Gamma\left(\frac{1}{2} - \nu + \frac{\xi}{2\kappa}\right)} (2\kappa R)^{2\nu} \tag{4.6}$$

where $\theta(x) = 1(0)$ for $x > 0$ $(x < 0)$. If we put $\xi = 0$ in (4.6) and use

$$\frac{\Gamma(1-2\nu)}{\Gamma(2\nu)} \frac{\Gamma\left(\frac{1}{2} + \nu\right)}{\Gamma\left(\frac{1}{2} - \nu\right)} = 2^{1-4\nu} \frac{\Gamma(1-\nu)}{\Gamma(\nu)} \tag{4.7}$$

we find that (4.6) is reduced to (2.6).

Let us first examine the case $\nu < \frac{1}{2}$. The method that we follow is the same as that used for the model of $H_0 + \frac{1}{2}m\omega^2 r^2$ examined in [1]. Imagine that $H_0$ has been already defined (before adding the Coulomb potential) and that it is characterized by parameter $\kappa_0$ as was done in section 2. The $H_0$ has a bound state with energy $-\hbar^2\kappa_0^2/2m$. When the

Coulomb potential is added to this $H_0$, the wavefunction $\phi(r)$ has to satisfy the following boundary condition:

$$G_0(\kappa_0 R) = G(\kappa R) \tag{4.8}$$

where $G_0(\kappa_0 R)$ is the $G(\kappa R)$ of (2.6) with $\kappa$ replaced by $\kappa_0$. By putting (4.6) into (4.7), we obtain

$$\frac{\Gamma\left(\frac{1}{2}+\nu+\frac{\xi}{2\kappa}\right)}{\Gamma\left(\frac{1}{2}-\nu+\frac{\xi}{2\kappa}\right)} = \frac{\Gamma\left(\frac{1}{2}+\nu\right)}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(\frac{\kappa_0}{\kappa}\right)^{2\nu}. \tag{4.9}$$

where $\xi = 0$ (4.9) means $\kappa = \kappa_0$ as it should. When $\xi \neq 0$, (4.9) determines possible values of $\kappa$. In the special case of $\kappa_0 = 0$, (4.9) leads to

$$\frac{1}{2}\left(1 - 2\nu + \frac{\xi}{\kappa}\right) = -(N-1) \qquad \kappa = -\frac{\xi}{N - \frac{1}{2} - \nu} \tag{4.10}$$

where $N = 1, 2, 3, \ldots$. Since $\kappa$ is positive by definition, (4.10) is valid only if $\xi < 0$, i.e. the Coulomb potential is attractive. The energy is given by $E = -\hbar^2\kappa^2/2m$ with $\kappa$ determined by (4.10). In this case the wavefunction is given by

$$\phi(r) = (\kappa r)^{-\nu} \mathrm{e}^{-\kappa r} L_N^{(-2\nu)}(2\kappa r) \tag{4.11}$$

where $L_N^{(-2\nu)}$ is the Laguerre polynomial. This $\phi(r)$ is singular at the origin.

If $H_0$ has no bound state, $G_0(\kappa_0 R)$ of (4.8) has to be replaced by $G_0(kR)$ of (2.2) with $k = \mathrm{i}\kappa$. In the special case of $\eta = 0$ is of interest. In this case $G_0(kR)$ for $kR \ll 1$ behaves like

$$G_0(kR) \simeq \nu - \frac{(kR)^2}{2(\nu+1)}. \tag{4.12}$$

Note that $G_0(kR) \to \nu$ as $R \to 0$. In order for $G(\kappa R)$ of (4.6) to satisfy the boundary condition, the coefficient of the term $(2\kappa R)^{2\nu}$ has to diverge. Thus we obtain

$$\frac{1}{2}\left(1 + 2\nu + \frac{\xi}{\kappa}\right) = -(N-1) \qquad \kappa = -\frac{\xi}{N - \frac{1}{2} + \nu}. \tag{4.13}$$

This determines the bound state spectrum for the Aharonov–Bohm system in the Coulomb potential. The wavefunctions for these states are regular at the origin, and again given by (4.11) but with the $\kappa$ of (4.13). If we put $\nu = 0$ in (4.13) we obtain the $S$-state energies for the two-dimensional hydrogen atom [10]. Hagen obtained (4.10) and (4.13). At this point let us emphasize that the irregular solutions of (4.10) and the regular solutions of (4.13) belong to two different SAEs of the Hamiltonian, i.e. two different Hamiltonians. These are only two of the examples of an infinite family of SAEs.

The special cases illustrated above may give the false impression that the possible values for $\kappa$ are limited. This is not so as we emphasized when we discussed the harmonic oscillator model in [1]. We can choose an SAE with an arbitrary value of $\kappa$. All that we have to do (and we can do) is to fine-tune the non-gauge potential of $r < R$ such that the boundary condition (2.4) with $G(\kappa R)$ of (4.6) is satisfied. As Hagen pointed out, $\kappa$ of (4.10) diverges as $\nu \to \frac{1}{2}$. Recall that (4.10) is for the special case in which we start with the SAE of $H_0$ with $\kappa = 0$. If we start with a SAE for $H_0$ such that there is no bound state, i.e. $\lambda$ of (2.10) is negative, then the $\kappa$ for the ground state will remain finite when $\nu \to \frac{1}{2}$. Let us repeat that the SAE of $H_0$ with $\kappa = 0$ for the spinless particle is equivalent to Hagen's model with the spin interaction of (3.1).

We now turn to the case of $\frac{1}{2} < \nu < 1$. The term proportional to $\xi$ of (4.6) prevents us from using (4.8). This does not mean that SAEs are impossible, rather we have to go back to the boundary condition (2.4). This time the $G(\kappa R)$ is the one given by (4.6) with the $\xi$-term. Equation (2.4) becomes

$$F(k_0R) = \frac{R}{J_n(k_0R)}\frac{\mathrm{d}J_n(k_0R)}{\mathrm{d}R} \simeq -\nu + \frac{\xi R}{2\nu - 1} + \frac{\Gamma(1-2\nu)}{\Gamma(2\nu)}\frac{\Gamma\left(\frac{1}{2}+\nu+\frac{\xi}{2\kappa}\right)}{\Gamma\left(\frac{1}{2}-\nu+\frac{\xi}{2\kappa}\right)}(2\kappa R)^{2\nu}. \tag{4.14}$$

We interpret that $k_0$ is a function of $R$ which is defined by (4.14) for a specified value of $\kappa$. We take the limit $R \to 0$. In doing so we keep $\kappa$ fixed while $k_0R$ approaches a certain finite (non-zero) value. Then obviously $k_0 \to \infty$; this requires a non-gauge potential like the square well potential with $D \to \infty$ as assumed in [1]. This is how we can obtain the SAE for the specified value of $\kappa$. We have also checked the deficiency index of the Hamiltonian and found it to be (1,1). This means that, as we mentioned in section 1, a one-parameter family of SAEs is allowed [4].

Let us comment on Hagen's calculation [5]. If we apply (4.14) to his model, $F(k_0R)$ takes the form

$$F(k_0R) = \frac{R}{J_n(k_0R)}\frac{\mathrm{d}J_n(k_0R)}{\mathrm{d}R} - \alpha s \tag{4.15}$$

where $k_0 = \mathrm{i}\kappa$ and the term $\alpha$ is due to the spin interaction (3.3) with $s = 1$. Suppose that $k_0R \to 0$ as $R \to 0$. Then we can expand $F(k_0R)$ as

$$F(k_0R) \simeq n - \frac{(k_0R)^2}{2(n+1)} - \alpha s. \tag{4.16}$$

Note that $k_0$ is a finite constant. It is clear that (4.14) with the $\xi$-term cannot be satisfied. Even if we assume a non-gauge potential for $r < R$, as long as the potential depth remains finite, (4.14) cannot be satisfied. This is essentially the situation which Hagen pointed out for the case of $\frac{1}{2} < \nu < 1$. As we stated in the preceding paragraph, SAEs are possible for $\frac{1}{2} < \nu < 1$, but it is crucial to have a non-gauge potential such that $D \to \infty$ as $R \to 0$.

## 5. The Dirac particle

Self-adjoint extensions of the Dirac Hamiltonian in the presence of an infinitely thin thread of magnetic flux has been the subject of several papers in recent years [2, 5, 6, 11]. Unlike the Schrödinger case which we have discussed in the preceding sections, SAEs of the Dirac Hamiltonian are possible only for $-1 < \nu < 0$. For $\nu > 0$, at least one of the components of the Dirac wavefunction becomes unnormalizable. A general framework of SAEs was described, for example, by Gerbert [11]. In [2] SAEs were discussed in the same manner as that of [1] by assuming a non-gauge potential within the flux $r < R$. The case in which a Coulomb potential is added was also discussed in [2]. Hagen [5] has recently criticized

[2]. His main objection is that the procedure described in [2] is unreliable because the Klein paradox may appear. In this section, however, we show that the difficulty that Hagen suggested does not actually occur. We also rectify some confusing points in the work presented in [2].

The Dirac equation in standard notation reads

$$(\alpha \cdot \Pi + \beta m + V)\Psi = E\Psi \qquad \Pi = p - eA. \tag{5.1}$$

Unlike in [2, 6, 7], we take (5.1) as the Dirac equation in three space dimensions with the understanding that $V$ is independent of $z$ and $p_z = 0$. The Dirac matrices $\alpha_x$, $\alpha_y$ and $\beta$ are $4 \times 4$. We write the four component spinor $\Psi$ as

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{5.2}$$

where each of $\phi$ and $\chi$ has two components. The potential $V(r)$ where $r = (x^2 + y^2)^{1/2}$ is the non-gauge potential which we may assume within the magnetic flux, i.e. for $r < R$. This is essentially what was denoted by $u_R$ in [2]. The potential $V(r)$ of (5.1) is the zeroth component of a Lorentz vector. We use units such that $c = \hbar = 1$ in this section.

Outside the flux, $r > R$, we assume that $V(r) = 0$. Then (5.1) can be reduced to

$$(\Pi^2 + m^2 - E^2)\phi = [(p - eA)^2 + m^2 - E^2]\phi = 0 \qquad r > R. \tag{5.3}$$

This can be further reduced to

$$\left(-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{\nu^2}{r^2}\right)\phi = k^2\phi \qquad \text{for } r > R \tag{5.4}$$

where $k^2 = E^2 - m^2$. When discussing a bound state we replace $k^2$ of (5.4) with $-\kappa^2$ such that $\kappa^2 = m^2 - E^2$. Equation (5.4) can be solved exactly in the same way as the Schrödinger equation. The lower component $\chi$ of $\Psi$ is related to $\phi$ by

$$\chi = \frac{-i}{m + E}\left(\frac{\partial}{\partial r} - \frac{\nu}{r}\right)\phi \qquad r > R. \tag{5.5}$$

Since $\nu$ appears in (5.5) we have to distinguish positive and negative $\nu$. The boundary condition at $r = R$ can be dealt with in terms of $\phi$ and $\chi$, or equivalently in terms of the logarithmic derivative of $\phi$, exactly as in the Schrödinger case.

Suppose there is a bound state. Then $\phi$ can be taken as

$$\phi = K_\nu(\kappa r) \tag{5.6}$$

where we have suppressed the factor which specifies the (up or down) spin state. $\chi$ is given by

$$\chi = \frac{-i\kappa}{m + E} K_{\nu+1}(\kappa r). \tag{5.7}$$

For $\kappa r \ll 1$, we find, apart from constant factors, $\phi \sim r^{-|\nu|}$ and $\chi \sim r^{-|\nu+1|}$. In order for both of $\phi$ and $\chi$ to be square integrable around the origin, we require that $-1 < \nu < 0$.

For a given value of $\alpha$, there is only one integer $n$ such that $-1 < \nu < 0$ is satisfied. The restriction $-1 < \nu < 0$ is assumed in this section unless otherwise stated.

The restriction $-1 < \nu < 0$ is well known of course. The reason we mention this is that we want to point out a very interesting aspect of Hagen's model calculation in this regard [5]. As Hagen emphasized, he does not require that the wavefunction is normalizable around the origin. By solving the Pauli equation for $\phi$ with the spin interaction of (3.2) and letting $r \to 0$, he finds that $|\nu| < 1$ which guarantees that $\phi$ is square integrable around the origin. We have confirmed that this remarkable feature appears also for the spin interactions considered by Bordag and Voropaev [6]. We suspect that this holds in the limit of $R \to 0$ for any arbitrary distribution of the magnetic field within the flux.

Inside the flux, if we assume $V(r)$ for $r < R$, solving the Dirac equation becomes a little more complicated. Equation (5.1) can be handled as coupled equations for $\phi$ and $\chi$ as was done in [2]. The boundary condition at $r = R$ is that $\phi$ and $\chi$ are both continuous. Alternatively, we can reduce (5.1) to an equation for the upper component,

$$\left[\Pi^2 - \frac{\mathrm{i}V'}{E+m-V}\frac{1}{r}(\sigma\cdot r)(\sigma\cdot\Pi) + m^2 - (E-V)^2\right]\phi = 0 \tag{5.8}$$

where $V' = \mathrm{d}V(r)/\mathrm{d}r$. $\Pi^2$ becomes

$$\Pi^2 = (p - eA)^2 - e\sigma_z B \qquad B = (\nabla \times A)_z. \tag{5.9}$$

The spin term with $\alpha s$ ((3.1) or (3.2)) derives from the term with $B$ of (5.9). We are interested in the situation that corresponds to $\alpha s < 0$. If $V(r)$ is a square well potential of radius $R$, $V'$ is of the form of $\delta(r - R)$. This and the spin term have to be carefully taken care of in matching the $\phi$ for $r < R$ and that for $r > R$. We do not delve into this aspect of the problem because the SAEs that we obtain after letting $R \to 0$ are, after all, insensitive to the details of the wavefunction for $r < R$. Let us rather focus on the question raised by Hagen regarding the Klein paradox.

For the discussion in the following it is sufficient to know that, if the potential for $r < R$ (the non-gauge potential plus the effective potential due to the magnetic field) is *attractive* and is a constant $-D < 0$, then $\phi$ for $r < R$ is of the form of

$$\phi = J_n(k_0 r) \qquad r < R \qquad k_0^2 = (E + D)^2 - m^2. \tag{5.10}$$

Here we assume a square well potential for simplicity but the specific form of the potential is not essential. We know that there is a bound state with $E = m$ or $\kappa = 0$ when $D = 0$; then $k_0 = 0$ [8]. Imagine that $D$ is increased starting with $D = 0$. Then the binding energy of the bound state increases, i.e. $\kappa > 0$ increases, and so does $k_0^2$.

Now suppose there is one and only one bound state with a certain value of $\kappa$. It is understood that the bound state energy $E$ is positive. This situation can be set up with an appropriate choice of the values of $R$ and $D$. Then $k_0^2 > 0$. Starting with this situation, decrease $R$ and increase $D$ in such a way that $\kappa$ is kept fixed. Then $k_0^2$ will increase and, in the limit $R \to 0$, we find that $D \to \infty$ and $k_0^2 \to \infty$. The value of $k_0^2$ does not oscillate in this process. This is how we can obtain an SAE of the Dirac Hamiltonian. In this limiting process no new bound states appear. Even when there are bound states in the beginning other than the one that we use for parametrizing the SAE, they all become unbound in the limit $r \to 0$. Generally, when there are a few bound states in a square well potential and when the radius $R$ is reduced, the level distance increases. This is basically why no new bound states are added in the limiting process of $R \to 0$. We have confirmed this by numerical experiments.

In [2], instead of the attractive non-gauge potential we assumed above, a *repulsive* non-gauge potential was assumed. That was unfortunate in the sense that the following unnecessary complication can occur. In order to have a positive energy bound state with a repulsive potential, its depth $D$ has to be much greater than $m$. For example in order to have a bound state with energy $E$ slightly less than $m$, we have to assume $D$ much larger than $2m$. In this case there can be more bound states whose energies are lower. Some of these energies may be negative. Nevertheless there is no difficulty in defining a SAE in the way we described in the preceding paragraph. In the limiting process $R \to 0$ by keeping the $E$ or $\kappa$ fixed for the bound state just below $m$, the $k_0^2$ remains always positive. This is because we start with $D > 2m$. There is no oscillation of the sign of $k_0^2$ which Hagen discussed. We have confirmed this also by numerical experiments. Let us add that, in [2] a repulsive non-gauge potential was thought of as a device for shielding the magnetic flux from the particle wavefunction. This idea is wrong. In fact we can have a bound state by a (very strong) repulsive potential as discussed above.

Let us briefly discuss the case in which a Coulomb potential $\xi/r$ is added. As shown in [2], $\phi$ behaves like $r^{\gamma-1/2}$ around the origin where $\gamma$ is defined by

$$\gamma = \pm[(\nu + \tfrac{1}{2})^2 - \xi^2]^{1/2}. \tag{5.11}$$

It is understood that $\xi^2 < \frac{1}{4}$. Since we are interested in irregular solutions, we choose the negative one from the double sign of (5.11). The square integrability of $\phi$ around the origin requires that $\gamma > \frac{1}{2}$, which leads to $\nu(\nu + 1) < \xi^2$, or

$$-\tfrac{1}{2} - \left(\tfrac{1}{4} + \xi^2\right)^{1/2} < \nu < \tfrac{1}{2} - \left(\tfrac{1}{4} + \xi^2\right)^{1/2}. \tag{5.12}$$

The mechanism behind this is simple. The Dirac equation with the Coulomb interaction $\xi/r$ can be written into a Schrödinger-like equation; see e.g. [12]. In that equation a term $-\xi^2/r^2$ appears. When this term is combined with the centrifugal potential term we find that $\nu + \frac{1}{2}$ is effectively replaced by $\gamma$. The wavefunction $\phi$ takes the same form as that for the corresponding non-relativistic case except that $\nu + \frac{1}{2}$ is replaced by $\gamma$. As discussed in [2] SAEs of the Dirac Hamiltonian are possible when (5.12) is satisfied. If $\alpha$ satisfies

$$|\alpha| < \tfrac{1}{2} + (\tfrac{1}{4} + \xi^2)^{1/2} \tag{5.13}$$

there are two integers for $n$ such that SAEs are possible.

## 6. Discussion

We have examined the relation between the SAEs of the Hamiltonian for a spinless particle in the Aharonov–Bohm field and the models considered by Hagen [5, 7] and by Bordag and Voropaev [6] for a spin-$\frac{1}{2}$ Schrödinger particle in the same environment. We also examined the case in which a Coulomb potential is added. We then proceeded to the Dirac particle and clarified a few confusing points of some recent papers [2, 5].

The model of [5, 7] and that of [6] are clear in their physical meaning. Mathematically, however, these models can be regarded as special cases of an infinite variety of possible SAEs. In this paper we have studied SAEs out of mathematical curiosity rather than from any specific physical motivation. However, SAEs can, in principle, be something more than a mathematical artifact. In [1] we discussed some situations, although unrelated to the Aharonov–Bohm problem, in which SAEs represent good approximations to real physical systems. When the gyromagnetic ratio $g$ of a spin-$\frac{1}{2}$ particle is greater than 2, a situation like an SAE will be realized as discussed by Bordag and Voropaev [6].

For the Dirac particle we considered only the Lorentz vector type for the non-gauge potential. Non-gauge potentials of Lorentz scalar type are also interesting. For example, a model such that the Dirac wavefunction is shielded from the flux could be constructed by means of a scalar-type interaction. In such a model, $\lambda$ of (2.10) can be used for parametrizing the extent of the shielding. In order to have complete suppression of the wavefunction on the surface of the magnetic flux, the scalar potential has to be infinitely strong and its radius larger than that of the flux. In [2] a repulsive non-gauge interaction of the Lorentz vector type was conceived as a means of suppressing the Dirac wavefunction in the magnetic flux region. That idea fails as we pointed out in section 5.

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